2020 B
March 18

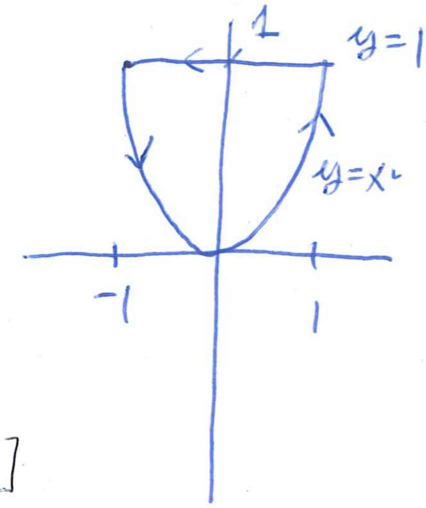
e.g. Find the circulation and flux of

$$\vec{F} = y\,\hat{i} + xy\,\hat{j}$$

around the closed curve C

when $C = C_1 + C_2$,

C_1 : $\vec{r}_1(x) = x\hat{i} + x^2\hat{j}$, $x \in [-1,1]$



C_2 : $\vec{r}_2(t) = (1,1) + t[(-1,1)-(1,1)]$

$= (1-2t, 1) = (1-2t)\hat{i} + \hat{j}$, $t \in [0,1]$.

circulation $= \oint_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r}$,

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{-1}^{1} \vec{F}(\vec{r}_1(x))\cdot \vec{r}_1{}'(x)\,dx$$

$$= \int_{-1}^{1} (x^2 \times 1 + x x^2 2x)\,dx$$

$$= \int_{-1}^{1} (x^2 + 2x^4)\,dx = \frac{22}{15}.$$

$$\int_{C_2} \vec{F}\cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}_2(t))\cdot \vec{r}_2{}'(t)\,dt$$

$$= \int_0^1 1\times(-2) + (1-2t)\times 0\ dt$$

$$= -2.$$

$\therefore$ circulation $= \frac{22}{15} - 2 = \frac{-8}{15}$ #

$$\text{flux} = \oint_C \vec{F}\cdot\hat{n}\,ds$$
$$= \int_{C_1} \vec{F}\cdot\hat{n}\,ds + \int_{C_2} \vec{F}\cdot\hat{n}\,ds$$
$$= \int_{C_1} (N\,dx - M\,dy) + \int_{C_2} (N\,dx - M\,dy).$$

On C_1,
$$N\,dx = N(\vec{r}_1(x))\,x'(x)\,dx$$
$$= x^3\,dx$$
$$M\,dy = x^2\,2x\,dx$$
$$\int_{C_1} N\,dx - M\,dy = \int_{-1}^{1} (x^3 - 2x^3)\,dx = 0.$$

On C_2
$$N\,dx = (1-2t)(1)(-2)\,dt$$
$$M\,dy = 1\times 0\,dt = 0$$
$$\int_{C_2} N\,dx - M\,dy = -2\int_0^1 (1-2t)\,dt = 0$$

$$\therefore \text{flux} = \int_{C_1} + \int_{C_2} (N\,dx - M\,dy) = 0.$$

e.g. Find the work done of $\vec{F} = \nabla f$, $f = xyz$, along a path from $A(-1,3,9)$ to $B(1,6,-4)$.

By thm,
$$\int_C \vec{F}\cdot d\vec{r} = f(B) - f(A)$$
$$= 1\times 6\times(-4) - (-1)\times 3\times 9$$
$$= 3 \ \#$$

A vector field is called independent of path if

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r}$$ if C_1 and C_2 have the same starting and ending points.

Theorem A vector field is conservative if and only if it is independent of path.

Pf. $\Rightarrow$ A previous thm shows that if $\vec{F} = \nabla f$, then

$$\int_C \vec{F}\cdot d\vec{r} = f(B) - f(A)$$

for any C from A to B. So $\vec{F}$ is indept of path.

$\Leftarrow$ Fix $A \in \Omega$ and let $B(x, y, z)$ be any point in Ω.

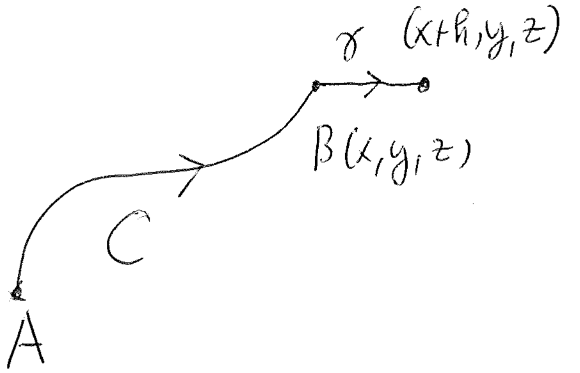
Define

$$f(x, y, z) = \int_C \vec{F}\cdot d\vec{r}$$

where C is any curve from A to $B(x,y,z)$. Since $\vec{F}$ is indept of path, f is indept of the choice of C. Claim:

$$\frac{\partial f}{\partial x}(x, y, z) = M(x, y, z).$$

Let $\vec{\gamma}(t) = (x+th, y, z)$, $t \in [0,1]$



$C+\gamma$ is a curve from A to $(x+h, y, z)$, so

$$f(x+th, y, z) = \int_{C+\gamma} \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot d\vec{r} + \int_\gamma \vec{F}\cdot d\vec{r}.$$

$$f(x+th,y,z)-f(x,y,z) = \int_\gamma \vec{F}\cdot d\vec{r}$$

$$= \int_0^1 M(x+th,y,z)\, h\, dt$$

$$\left(\begin{array}{l} \vec{\gamma}(t) = (x+th, y, z) \\ \vec{\gamma}'(t) = (h,0,0) \end{array} \right)$$

$$\frac{f(x+th,y,z)-f(x,y,z)}{h} = \int_0^1 M(x+th,y,z)\, dt$$

$$\lim_{h\to 0} \frac{f(x+th,y,z)-f(x,y,z)}{h} = \int_0^1 M(x,y,z)\, dt$$
$$= M(x,y,z).$$

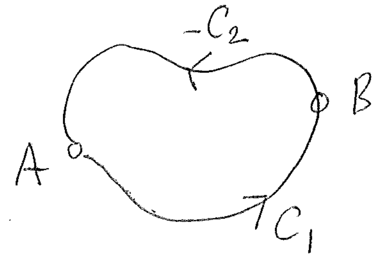
Similarly, one can show

$$\frac{\partial f}{\partial y} = N, \quad \frac{\partial f}{\partial z} = P. \quad \#$$

<u>Remark</u> A v.f. is called to have the "loop property" if

$$\oint_C \vec{F}\cdot d\vec{r} = 0$$

for every closed curve C. Obviously, this is equivalent to $\vec{F}$ being indept of path. Since any closed curve can be expressed as $C = C_1 - C_2$ for C_1, C_2 having the same starting and ending points



Q. How to check if $\vec{F}$ is conservative ?

We have a simple necessary condition.

Theorem $\vec{F} = M\hat{i} + N\hat{j} + P\hat{k}$ is conservative. Then

$$\frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \quad \frac{\partial N}{\partial z} = \frac{\partial P}{\partial y}.$$

when $\vec{F} = M\hat{i} + N\hat{j}$ in the plane,

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Pf. If $\vec{F} = \nabla f$. then

$$\frac{\partial M}{\partial z} = \frac{\partial}{\partial z}\frac{\partial f}{\partial x} = \frac{\partial^2 f}{\partial z \partial x}, \quad \frac{\partial P}{\partial x} = \frac{\partial}{\partial x}\frac{\partial f}{\partial z} = \frac{\partial^2 f}{\partial x \partial z}$$

$$\therefore \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x} \quad \#$$

Similar, for others.

e.g. Show that $\vec{F} = (2x-3)\hat{i} - z\hat{j} + \cos z\,\hat{k}$ is not conservative.

$$\frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}$$

$$\frac{\partial M}{\partial y} = 0 = \frac{\partial N}{\partial x}$$

but $\frac{\partial N}{\partial z} = -1 \neq \frac{\partial P}{\partial y} = 0$.

But, it could happen that all 3 conditions are satisfied, ~~but~~ the v.f. is not conservative.

These conditions are necessary but not sufficient for the existence of a potential!

e.g. Show that

$$\vec{F} = \frac{-y}{x^2+y^2}\hat{i} + \frac{x}{x^2+y^2}\hat{j} + 0\hat{k}$$

Satisfies the 3 conditions but is NOT Conservative.

$$M = \frac{-y}{x^2+y^2}, \quad N = \frac{x}{x^2+y^2}, \quad P = 0$$

$$\frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \quad \checkmark$$

$$\frac{\partial M}{\partial y} = \frac{-x^2+y^2}{(x^2+y^2)^2} = \frac{\partial N}{\partial x}, \quad \checkmark$$

$$\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z} \quad \checkmark$$

On the other hand, $\vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + 0\,\hat{k}$, $t \in [0, \pi]$, is the circle on the xy-plane, a closed curve.

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C M\,dx + N\,dy + P\,dz$$

$$= \int_0^{2\pi} \left[\frac{-\sin t}{\cos^2 t + \sin^2 t}(-\sin t) + \frac{\cos t}{\cos^2 t + \sin^2 t}(\cos t) + 0 \right] dt$$

$$= 2\pi \neq 0.$$

Since $\vec{F}$ does not satisfy the loop property, $\vec{F}$ is not conservative!

Now, the potential can be found by integration.

e.g. Find the potential (if exists) for

$$\vec{F} = (e^x \cos y + yz)\,\hat{i} + (xz - e^x \sin y)\,\hat{j} + (xy + z)\,\hat{k}.$$

Step 1 Checking $\frac{\partial M}{\partial z} = y$, $\frac{\partial P}{\partial x} = y$ ✓

$$\frac{\partial M}{\partial y} = -e^x \sin y + z, \quad \frac{\partial N}{\partial x} = z - e^x \sin y \quad \checkmark$$

$$\frac{\partial N}{\partial z} = x, \quad \frac{\partial P}{\partial y} = x \quad \checkmark$$

Step 2 Integration.

$$\frac{\partial f}{\partial x} = M = e^x \cos y + yz \Rightarrow$$

$f(x,y,z) = e^{x}\cos y + xyz + h(y,z)$ for some h,

$$\frac{\partial f}{\partial y}(x,y,z) = -e^{x}\sin y + xz + \frac{\partial h}{\partial y}$$

$$= N = xz - e^{x}\sin y$$

$$\Rightarrow \frac{\partial h}{\partial y} = 0, \text{ ie, } h(y,z) = \varphi(z) \text{ for som } \varphi.$$

$$\frac{\partial f}{\partial z} = xy + \varphi'(z) = xy + z \Rightarrow \varphi'(z) = z$$

$$\varphi(z) = \frac{z^2}{2} + C$$

Conclusin : $f(x,y,z) = e^{x}\cos y + xyz + \frac{z^2}{2} + C$ is the potential.